Physics 319
Classical Mechanics

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Lecture 25

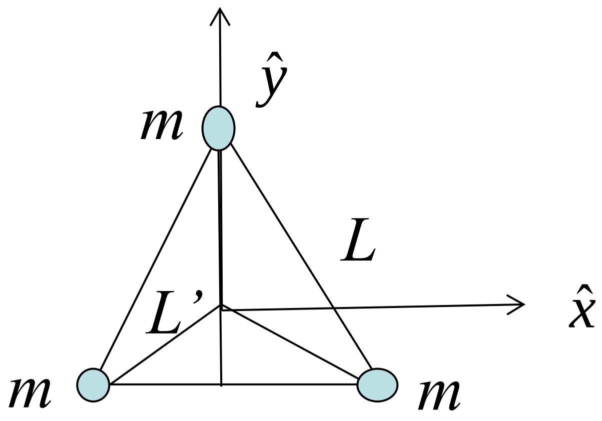
Inertia Tensor for Equilateral Triangle

$$I_{xx} = 2m\frac{L^2}{12} + m\frac{L^2}{3} = mL^2/2$$

$$I_{yy} = 2m\left(\frac{L}{2}\right)^2 + 0 = mL^2/2$$

$$I_{zz} = 3mL'^2 = 3mL^2/3 = mL^2$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$



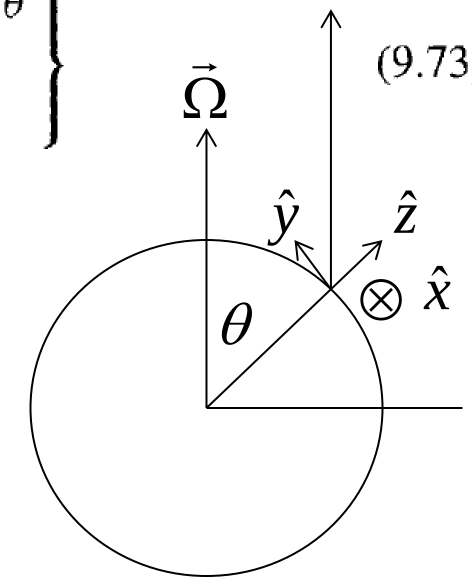
$$L'\frac{\sqrt{3}}{2} = \frac{L}{2} \rightarrow L' = \frac{L}{\sqrt{3}}$$

- Principal Axes $\hat{x}, \hat{y}, \hat{z}$
- Tricks
  - Lay out axes along symmetry axes
  - Plane masses should be on $z = 0$
  - Memorize elements

Problem 9.26

9.26 ★★ In Section 9.8, we used a method of successive aproximations to find the orbit of an object that is dropped from rest, correct to first order in the earth's angular velocity Ω. Show in the same way that if an object is thrown with initial velocity $\mathbf{v}_o$ from a point O on the earth's surface at colatitude θ, then to first order in Ω its orbit is

$$\left.\begin{aligned} x &= v_{xo}t + \Omega(v_{yo}\cos\theta - v_{zo}\sin\theta)t^2 + \tfrac{1}{3}\Omega g t^3 \sin\theta \\ y &= v_{yo}t - \Omega(v_{xo}\cos\theta)t^2 \\ z &= v_{zo}t - \tfrac{1}{2}gt^2 + \Omega(v_{xo}\sin\theta)t^2. \end{aligned}\right\} \quad (9.73)$$



- Zeroeth approximation

$$\ddot{x} = 0$$

$$\ddot{y} = 0$$

$$\ddot{z} = -g$$

$$x(t) = x_0 + v_{0x}t$$

$$y(t) = y_0 + v_{0y}t$$

$$z(t) = z_0 + v_{0z}t - gt^2/2$$

$$\vec{\Omega} = \Omega(\sin\theta\hat{y} + \cos\theta\hat{z})$$

$$\Omega = \frac{2\pi}{(24)(3600)}\ \text{sec}^{-1}$$

First Approximation

- Include Coriolis force

$$\ddot{x} = F_{Cor,x} / m$$

$$\ddot{y} = F_{Cor,y} / m$$

$$\ddot{z} = -g + F_{Cor,z} / m$$

$$\frac{\vec{F}_{Cor}}{m} = 2\Omega \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & \sin\theta & \cos\theta \end{vmatrix} = 2\Omega \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_{x0} & v_{y0} & v_{z0} - gt \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

$$\ddot{x} = 2\Omega\left(v_{y0}\cos\theta - \left(v_{z0} - gt\right)\sin\theta\right)$$

$$\ddot{y} = 2\Omega\left(-v_{x0}\cos\theta\right)$$

$$\ddot{z} = -g + 2\Omega\left(v_{x0}\sin\theta\right)$$

Result

- Integrate twice and put the origin at $t = 0$ location

$$\dot{x} = v_{0x} + 2\Omega\left(v_{y0}\cos\theta t - \left(v_{z0}t - gt^2/2\right)\sin\theta\right)$$

$$\dot{y} = v_{0y} - 2\Omega\left(v_{x0}\cos\theta t\right)$$

$$\dot{z} = v_{0z} - gt + 2\Omega\left(v_{x0}\sin\theta t\right)$$

$$x = v_{0x}t + \Omega\left(v_{y0}\cos\theta - v_{z0}\sin\theta\right)t^2 + \Omega g t^3 \sin\theta / 3$$

$$y = v_{0y}t - \Omega\left(v_{x0}\cos\theta\right)t^2$$

$$z = v_{0z}t - gt^2/2 + \Omega\left(v_{x0}\sin\theta\right)t^2$$

Equations on Orbits

- Ellipse in polar coordinates

$$r(\phi) = \frac{c}{1 + \epsilon\cos(\phi - \delta)}.$$

- Aphelion and Perihelion distance

$$r_{\mathrm{max}} = \frac{c}{1+\varepsilon} \qquad r_{\mathrm{min}} = \frac{c}{1+\varepsilon}$$

- Semimajor axis, semiminor ellipse, and focus dimensions

$$a = \frac{c}{1-\epsilon^2}, \qquad b = \frac{c}{\sqrt{1-\epsilon^2}}, \qquad \text{and} \qquad d = a\epsilon.$$

Halley Comet Example

Halley's comet, named for the English astronomer Edmund Halley (1656–1742), follows a very eccentric orbit with $\epsilon = 0.967$. At closest approach (the perihelion) the comet is 0.59 AU from the sun, fairly close to the orbit of Mercury. (The AU or astronomical unit is the mean distance of the earth from the sun, about 1.5×10^8 km.) What is the comet's greatest distance from the sun, that is, the distance of the aphelion?

$$\frac{r_{\max}}{r_{\min}} = \frac{c}{1-\varepsilon}\frac{1+\varepsilon}{c} = \frac{1.967}{0.033} 0.59 \text{ AU} = 35 \text{ AU}$$